Series 1

Notation:

- $\mathbb{R}_+ := (0, +\infty)$.
- For any square matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ for $n \in \mathbb{N}$, we denote by $|\boldsymbol{A}|$ its determinant.
- For $n, m \in \mathbb{N} \cup \{0\}$ let

  $C^n_m([0,1]\times[0,1]) := \left\{u : [0,1] \times [0,1] \to \mathbb{R} : u, \partial_x u, \partial_x^2 u, \dots, \partial_x^n u, \partial_t u, \dots, \partial_t^m u \in C\left([0,1] \times [0,1]\right)\right\}.$

1. Option Pricing and the Black Scholes equation

a) Let X, Z be real-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G} \subseteq \mathcal{F}$ be a sub-σ-algebra such that X is $\mathcal{G}$-measurable and Z is independent of $\mathcal{G}$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be Borel-measurable and bounded. Define $h(x) := \mathbb{E}[f(x, Z)]$ for all $x \in \mathbb{R}$.

(i) Show that

$$\mathbb{E}[f(X, Z) \mid \mathcal{G}] = h(X) \quad \text{a.s.}$$

in the special case where f is the indicator of a measurable rectangle, i.e.

$$f(x, y) = \mathbf{1}_{A\times B}(x, y) = \mathbf{1}_A(x)\mathbf{1}_B(y), \quad A, B \in \mathcal{B}(\mathbb{R}).$$

(ii) Using (i), extend the identity

$$\mathbb{E}[f(X, Z) \mid \mathcal{G}] = h(X) \quad \text{a.s.}$$

to all bounded Borel-measurable functions $f : \mathbb{R}^2 \to \mathbb{R}$.

Hint: You may use, without proof, that for every bounded Borel-measurable function $f : \mathbb{R}^2 \to \mathbb{R}$ there exists a sequence of simple functions of the form

$$f^n(x, y) = \sum_{i=1}^{k_n} a_i^{(n)} \mathbf{1}_{A_i^{(n)}\times B_i^{(n)}}(x, y),$$

with $a_i^{(n)} \in \mathbb{R}$ and $A_i^{(n)}, B_i^{(n)} \in \mathcal{B}(\mathbb{R})$, such that $f^n(x, y) \to f(x, y)$ and $|f^n(x, y)| \le |f(x, y)|$ for all $(x, y) \in \mathbb{R}^2$.

b) Let W be a one-dimensional Brownian motion and let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be a filtration satisfying the usual conditions. For $S_0, r, \sigma > 0$, define the Black–Scholes stock price process

$$S_t = S_0 \exp\Big(\big(r - \tfrac{1}{2}\sigma^2\big)t + \sigma W_t\Big), \qquad t \in [0, T].$$

Let $g : \mathbb{R}_+ \to \mathbb{R}$ be Borel-measurable such that $\mathbb{E}[|g(S_T)|] < \infty$. Recall that the no-arbitrage price at time $t \in [0, T]$ of a European claim with payoff $g(S_T)$ at maturity T is

$$V_t = e^{-r(T-t)}\, \mathbb{E}[g(S_T) \mid \mathcal{F}_t], \quad t \in [0, T].$$

(i) Show that for a European *digital call* option with payoff $g(S_T) = \mathbf{1}_{\{S_T \ge K\}}$ for a strike $K > 0$, we have

$$V_t = v(t, S_t), \tag{1}$$

where for $(t,s) \in [0,T) \times \mathbb{R}_+$,

$$v(t,s) = e^{-r(T-t)}\, \Phi\left(\frac{\ln(s/K) + \big(r - \frac{1}{2}\sigma^2\big)(T-t)}{\sigma\sqrt{T-t}}\right), \tag{2}$$

and Φ denotes the cumulative distribution function of a standard normal random variable given by

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\,\mathrm{d}y, \quad x \in \mathbb{R}.$$

(ii) Show that v given by (2) satisfies the Black–Scholes PDE

$$\partial_t v(t,s) + \frac{1}{2}\sigma^2 s^2\, \partial^2_{ss} v(t,s) + rs\, \partial_s v(t,s) - rv(t,s) = 0, \tag{3}$$

for $(t,s) \in [0,T) \times \mathbb{R}_+$.

(iii) Compute the limit

$$\lim_{t \uparrow T} v(t,s)$$

for each $s \in \mathbb{R}_+$. What do you observe?

c) Show that the Black Scholes equation (3) can be reduced to the classical heat equation

$$\partial_t u - \partial^2_{xx} u = 0,$$

by an appropriate change of variables. Specify the transformation and derive the resulting PDE.

Hint: Write $v(t, S_t)$ as a function of the Brownian motion and use Ito's formula.

2. Finite-Difference method for the heat equation

Consider the following heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} &= 0 & \text{on } (0,1) \times (0,1)\,, \\ u(x,0) &= u_0(x) & \forall x \in (0,1)\,, \\ u(0,t) = \dfrac{\partial u}{\partial x}(1,t) = &= 0 & \forall t \in (0,1)\,. \end{cases} \tag{4}$$

We assume that $u_0 \in C^2([0,1])$.

a) Show that for any solution $u \in C^2_1([0,1] \times [0,1])$ to (4), the function $f(t) := \int_0^1 u(x,t)^2 dx,\ t \in [0,1]$ is non-increasing on $[0,1]$. Show furthermore that for any $u_0 \in C^2([0,1])$, there exists at most one smooth solution $u \in C^2_1([0,1] \times [0,1])$ to (4).

Let $N, M \in \mathbb{N}$. For all $i = 0, \dots, N$ and $m = 0, \dots, M$ with $N, M \in \mathbb{N}$, we seek approximations of the values

$$u(x_i, t_m) := u(ih, mk)$$

on the spatio-temporal mesh $\{(ih, mk)\}_{i=0,\dots,N, m=0,\dots,M}$ where $h = \frac{1}{N}$ is the mesh width and $k = \frac{1}{M}$ is the time step. The approximations are denoted by u_i^m, and are obtained by solving the following linear system of equations iteratively for $m \in \{0, \dots, M-1\}$

$$\begin{cases} \dfrac{u_i^{m+1} - u_i^m}{k} - \dfrac{u_{i-1}^m - 2u_i^m + u_{i+1}^m}{h^2} &= 0\,, \quad \text{for } 0 < i \le N\,, \\[2ex] \dfrac{-u_{N-1}^{m+1} + u_{N+1}^{m+1}}{2h} &= 0\,, \\[2ex] u_0^{m+1} &= 0\,, \end{cases} \tag{5}$$

with initial values $u_i^0 = u_0(ih)$ ("explicit method"). Take note that u_{N+1}^m for $m \in \{0, \dots, M\}$ are fictitious values introduced to suitably treat the Neumann boundary condition.

b) Assume that $u \in C_2^4([0,1] \times [0,1])$. Show that

$$\begin{aligned} \frac{\partial u}{\partial t}|_{(x,t)=(x_i,t_m)} &= \frac{u_i^{m+1} - u_i^m}{k} + O(k)\,, \\ \frac{\partial u}{\partial x}|_{(x,t)=(x_N,t_m)} &= \frac{u_{N+1}^m - u_{N-1}^m}{2h} + O(h^2), \\ \frac{\partial^2 u}{\partial x^2}|_{(x,t)=(x_i,t_m)} &= \frac{u_{i+1}^m - 2u_i^m + u_{i-1}^m}{h^2} + O(h^2)\,. \end{aligned}$$

c) After eliminating the variables u_0^m and u_{N+1}^m, rewrite the system (5) in the form

$$\underline{u}^{m+1} = \boldsymbol{C}\underline{u}^m\,, \quad \text{where } \underline{u}^m := \begin{pmatrix} u_1^m \\ u_2^m \\ \cdots \\ u_N^m \end{pmatrix} \in \mathbb{R}^N\,.$$

Show that $\boldsymbol{C}$ is given by $\boldsymbol{C} = \boldsymbol{I} + \nu\boldsymbol{G}$ for some coefficient $\nu > 0$ where $\boldsymbol{I}$ is the $N \times N$ identity matrix and

$$\boldsymbol{G} = \begin{pmatrix} -2 & 1 & 0 & \dots & \dots & 0 \\ 1 & -2 & 1 & \ddots & (0) & \vdots \\ 0 & 1 & -2 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & (0) & \ddots & 1 & -2 & 1 \\ 0 & \dots & \dots & 0 & 2 & -2 \end{pmatrix}.$$

d) Consider an alternative way to discretize (4) which is described below ("implicit method").

$$\begin{cases} \dfrac{u_i^{m+1} - u_i^m}{k} - \dfrac{u_{i-1}^{m+1} - 2u_i^{m+1} + u_{i+1}^{m+1}}{h^2} &= 0\,, \quad \text{for } 0 < i \le N\,, \\[2ex] \dfrac{-u_{N-1}^{m+1} + u_{N+1}^{m+1}}{2h} &= 0\,, \\[2ex] u_0^{m+1} &= 0\,, \end{cases} \tag{6}$$

After eliminating the variables u_0^{m+1} and u_{N+1}^{m+1}, rewrite the system (5) in the form

$$\boldsymbol{C}\underline{u}^{m+1} = \underline{u}^m\,, \quad \text{where } \underline{u}^m := \begin{pmatrix} u_1^m \\ u_1^m \\ \cdots \\ u_N^m \end{pmatrix}.$$

Show that $\boldsymbol{C}$ is given by $\boldsymbol{C} = \boldsymbol{I} - \nu\boldsymbol{G}$ for some coefficient $\nu > 0$.

3. Implementation in Python

We implement Finite Difference Method to (4) using Python. From now on we set $u_0(x) = \sin\left(\frac{\pi}{2}x\right)$. All the coding tasks in this exercise shall be finished in the template `1_exercise3_template.py`.

a) Verify that $u(x,t) = e^{-\frac{\pi^2}{4}t}\sin\left(\frac{\pi}{2}x\right)$ is the solution to (4). Use this result to complete the Python function "`exact_solution_at_1(x)`", which receives a vector of grid points and computes a vector containing the value of $u(x,1)$ at these points.

b) Complete the Python functions "`eulerexplicit(N,M)`" and "`eulerimplicit(N,M)`" which compute the numerical approximation of $u(x,1)$ using the numerical scheme (5) and (6) in matrix form. The parameters of both functions correspond to $N, M \in \mathbb{N}$ as described in Exercise 2 and the return value u_M shall be the numerical solution of u at $t = 1$, which is an array with N elements (not including the value at $x = 0$).

c) Test both functions with the template using the parameters $N = 2^l$ and $M = 2 \times 4^l$ with $l = \{2,3,4,5,6\}$ and generate the convergence plot with respect to $h^2 + k$. To do that, follow the comments in the code and modify the block "error analysis".

d) Perform the same test as in **c**) using the parameters $N = 2^l$ and $M = 4^l$ with $l = \{2,3,4,5,6\}$ and report on the performances of the two methods. What happens to the discrete L^2 error of the numerical solution from scheme (5)? Check the error stored in the array "`l2errorexplicit`".

Due: Wednesday, March 4th, at 12:00.